Effect of an electric field on heat transfer in a paraelectric gas

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This paper deals theoretically with some aspects of the influence of a non-homogeneous electric field on the laminar convective motion and heat transfer in paraelectric gas, i.e. a gas consisting of molecules having a permanent electric dipole moment. It is found that, due to the variation of the dielectric susceptibility with temperature, the electric field produces an electrical buoyancy force. Convective velocities and heat transfer in the gas near a heated surface are found to be increased or decreased according as the electrical buoyancy force acts with or in opposition to the net force of the existing pressure gradient and gravitational buoyancy force.

The equations of motion for a paraelectric gas in the presence of an electric field are derived in a simplified form by the use of approximations similar to those of Boussinesq (1903). An exact solution of these equations is presented for the problem of laminar convection flow, under a pressure gradient, between vertical concentric cylinders which are maintained at different electrostatic potentials and whose wall temperatures decrease uniformly with increasing height. Here the electric field induces a heated down-flow to be superimposed on the existing cooled up-flow (or heated down-flow).

Boundary-layer equations are also derived for the laminar convective motion due to a heated charged sphere. These equations are solved by an approximate method due to Squire (1938).

1. Introduction

Senftleben & Braun (1936) have carried out an experimental investigation on the effect of an electric field on natural convection flow in gases. They discovered that if an electrostatic potential difference is applied between a heated horizontal wire and a concentric cylinder maintained at a lower temperature, the heat transfer from the wire is increased when the annulus is filled with a paraelectric gas, i.e. a gas consisting of molecules having a permanent electric dipole moment. The explanation is as follows: in the presence of the radial electric field the gas will experience an additional body force. This force acts radially inwards and depends on the strength of the electric field and the dielectric susceptibility, which decreases with increasing temperature. Consequently an element of hot gas will experience a lesser force than an element of cold gas in the same position. Now if the wire were at the same temperature as the surrounding gas the electrical and gravitational body forces would be in equilibrium with the hydrostatic pressure and so no flow would ensue. However, if the wire is heated the resulting temperature gradient near the wire causes a defect in the radial electrical body force. This is superimposed on the existing defect in the vertical gravitational body force and so there is an increase in the convection of heated gas away from the wire and so an increase in heat transfer from the wire.

In a later paper Kronig & Schwarz (1949) analysed the measurements of the above experiment and demonstrated how the results for different gases could be correlated on the introduction of a new dimensionless group, which plays a similar role in the effect of the electric forces as the Grashof number in the effect of the gravitational buoyancy force. The effect of an electric field on heat transfer has been used in practice for the continuous analysis of gas mixtures containing a paraelectric component (see Kronig 1942 and Schwarz 1949).

In the present paper some theoretical aspects of the influence of a non-homogeneous electric field on heat transfer rates in a paraelectric gas are discussed. First, in §2 the equations relating to the laminar convective motion of the gas are considered and simplified. In §3 an exact solution of these equations is obtained for the flow and heat transfer of a paraelectric gas, under a pressure gradient, between vertical concentric cylinders which are maintained at different electrostatic potentials and whose wall temperatures decrease uniformly with increasing height. In §4 the boundary-layer equations for the convective motion due to a heated charged sphere are derived and solved by an approximate method.

2. The equations of steady motion

Following Boussinesq (1903) the equations for steady motion of a paraelectric gas are simplified by assuming that:

(i) The temperature difference $T-T_0$ is small compared with the absolute temperature T_0, which is usually taken as the ambient temperature of the gas.

(ii) All physical constants of the gas are independent of the temperature and allowance is made for variations in density and dielectric susceptibility only in the calculation of the body forces.

(iii) The fluid is incompressible and viscous heat dissipation may be neglected.

Then the governing equations of motion and the electric field are:

$$\operatorname{div} \mathbf{v} = 0, \tag{2.1}$$

$$-\rho\mathbf{v}\times\operatorname{curl}\mathbf{v} = \mathbf{F}-\operatorname{grad}(p+\tfrac{1}{2}\rho\mathbf{v}^2)-\rho\nu\operatorname{curl}\operatorname{curl}\mathbf{v}, \tag{2.2}$$

$$\mathbf{v}\,.\operatorname{grad} T = k\operatorname{div}\operatorname{grad} T, \tag{2.3}$$

$$\operatorname{curl}\mathbf{E} = 0, \tag{2.4}$$

and

$$\operatorname{div}(1+4\pi\chi)\,\mathbf{E} = 0. \tag{2.5}$$

Here $\mathbf{v}$ is the velocity, ρ the density, $\mathbf{F}$ the body force, p the pressure, ν the kinematic viscosity, T the temperature, k the thermal diffusivity, $\mathbf{E}$ the electric field and χ the dielectric susceptibility.

In the case of natural convection, with gravity the only force acting, the gravitational body force per unit volume of gas is

$$\mathbf{F}_g = \rho\mathbf{g}, \tag{2.6}$$

and the appropriate equation of state is

$$\frac{\rho-\rho_0}{\rho_0} = -\frac{T-T_0}{T_0}. \tag{2.7}$$

However, if the gas is paraelectric there will be an additional force due to the applied electric field **E**. The electrical body force per unit volume of gas is (see Landau & Lifshitz 1960, p. 64)

$$\mathbf{F}_e = \tfrac{1}{2}\operatorname{grad}\{\mathbf{E}^2\rho(\partial\chi/\partial\rho)_T\} - \tfrac{1}{2}\mathbf{E}^2\operatorname{grad}\chi. \tag{2.8}$$

In order to calculate $\mathbf{F}_e$ a knowledge of the dependence of χ on ρ and T is required. This is given by Debye's theory of electric polarization, according to which for a dielectric gas (see Loeb 1927, Ch. 10)

$$\chi = \frac{N\rho}{M}\left(\sigma + \frac{\mu^2}{3k_B T}\right). \tag{2.9}$$

Here N is Avogadro's number, M the molecular weight, σ the polarizability of a gas molecule, μ the dipole moment of a gas molecule and k_B the Boltzmann constant. From Debye's law it follows that $\rho(\partial\chi/\partial\rho)_T = \chi$ and expression (2.8) becomes

$$\mathbf{F}_e = \tfrac{1}{2}\chi\operatorname{grad}\mathbf{E}^2. \tag{2.10}$$

This expression can be further simplified with assumption (i); thus

$$\chi(\rho, T) = \chi_0 + (\partial\chi/\partial\rho)_{T_0}(\rho-\rho_0) + (\partial\chi/\partial T)_{\rho_0}(T-T_0) + \ldots, \tag{2.11}$$

where the suffix 0 signifies some reference condition of the system. Since

$$(\rho-\rho_0)/\rho_0 = -(T-T_0)/T_0 \quad \text{and} \quad (\partial\chi/\partial\rho)_{T_0} = \chi_0/\rho_0,$$

it follows that

$$(\chi-\chi_0)/\rho_0 m = -(T-T_0)/T_0, \tag{2.12}$$

where

$$m = \frac{N}{M}\left(\sigma + \frac{2\mu^2}{3k_B T_0}\right) \quad \text{and} \quad \chi_0 = \frac{N}{M}\rho_0\left(\sigma + \frac{\mu^2}{3k_B T_0}\right). \tag{2.13}$$

The total body force **F** acting on unit volume of gas is then

$$\mathbf{F} = \mathbf{F}_g + \mathbf{F}_e = \rho\mathbf{g} + \tfrac{1}{2}\chi\operatorname{grad}\mathbf{E}^2. \tag{2.14}$$

It remains now to discuss the determination of the electric field **E**. Since the dielectric susceptibility of the gas is very small (e.g. $4\pi\chi = 7{\cdot}2\times10^{-3}$ for ammonia gas at N.T.P.), assumption (i) implies that equation (2.5) can be approximated to simply by

$$\operatorname{div}\mathbf{E} = 0. \tag{2.15}$$

It then follows from (2.4) and (2.15) that

$$\mathbf{E} = -\operatorname{grad}\phi \quad \text{and} \quad \nabla^2\phi = 0, \tag{2.16}$$

where ϕ is the electrostatic potential. Equations (2.15) and (2.16) are equivalent to the statement that the electric field is uninfluenced by the non-uniform motion of the gas.

With the aid of equations (2.14), (2.16), (2.7) and (2.12) the combined electrical and gravitational body force may be determined for any system. In fact the electrical body force is analogous to the gravitational body force. This can be seen more clearly on consideration of the defect in these forces produced by heating the gas. The gravitational buoyancy force per unit volume of gas is

$$(\mathbf{F}-\mathbf{F}_0)_g = (\rho-\rho_0)\,\mathbf{g} = -\rho_0\{(T-T_0)/T_0\}\,\mathbf{g}, \tag{2.17}$$

and the electrical buoyancy force per unit volume of gas is

$$(\mathbf{F}-\mathbf{F}_0)_e = \tfrac{1}{2}(\chi-\chi_0)\,\mathrm{grad}\,\mathbf{E}^2 = -\tfrac{1}{2}\rho_0 m\{(T-T_0)/T_0\}\,\mathrm{grad}\,\mathbf{E}^2. \tag{2.18}$$

Furthermore, it follows from (2.17) and (2.18) that a non-homogeneous electric field $\mathbf{E}$ will produce, at any position, a substantial electrical buoyancy effect if

$$\tfrac{1}{2}\,\mathrm{grad}\,(m\mathbf{E}^2) \gg \mathbf{g}, \tag{2.19}$$

a condition which can be satisfied in practice provided the gas is paraelectric.

Thus the equations relating to the steady convective motion of a heated paraelectric gas in the presence of an electric field are (2.1), (2.2), (2.3), (2.14), (2.7), (2.12) and (2.16) subject to certain boundary conditions to be stated as required. Note that all physical constants appearing in these equations must be evaluated at the reference temperature T_0. In the following, the e.s.u. and c.g.s. systems of units are adopted.

3. The influence of a radial electric field on the cooled up-flow (or heated down-flow) of a paraelectric gas between vertical concentric cylinders

Consider the steady fully developed laminar flow, under a pressure gradient, of a paraelectric gas between vertical concentric cylinders of radius a and b cm respectively ($b > a$), which are maintained at a uniform temperature gradient (τ/a) °C/cm in the direction of the axis. The system will be referred to cylindrical polar co-ordinates (r, ϕ, x) and due to axial symmetry will be independent of ϕ. The cylinders are charged to potentials V_a and V_b volts resulting in the electrostatic field

$$E_r = E_s a/r, \quad E_x = E_\phi = 0, \tag{3.1}$$

where $E_s = (V_a - V_b)/300\,a\log(b/a)$ e.s.u. The temperature of the walls is

$$T_w = T_0 - \tau x/a \quad (\tau > 0). \tag{3.2}$$

A similarity solution of the basic equations of motion is possible provided

$$\mathbf{v} = \{u(r), 0, 0\}, \quad T = T_w + \theta(r). \tag{3.3}$$

The equation of continuity (2.1) is identically satisfied, and the momentum equation (2.2) and thermal energy equation (2.3) become

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u - g, \tag{3.4}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial r} - \frac{\chi E_s^2 a^2}{\rho r^3}, \tag{3.5}$$

$$0 = \nabla^2\theta + (\tau/ka)\,u, \tag{3.6}$$

where now

$$\nabla^2 = \frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right).$$

If E_s is zero equations (3.4) and (3.5) imply that the pressure will vary only in the direction of flow. The resulting similarity solution relates to the combined free and forced convection for a fully developed cooled up-flow (or heated down-flow). A detailed discussion of these equations has been given by Morton (1960) for the combined free and forced convection in a uniformly cooled (or heated) vertical tube.

Now if E_s is non-zero equation (3.5) implies a radial pressure gradient. Moreover, as χ decreases linearly with increasing height (see equations (2.12), (3.2) and (3.3)) there will be an induced axial pressure gradient due to the electric field. More precisely, on using expressions (2.7) and (3.3), equation (3.4) is

$$\mu\nabla^2 u + \beta g\rho_w\theta = \partial p/\partial x + \rho_w g, \tag{3.7}$$

where $\beta = 1/T_0$. Equation (3.7) implies that

$$\partial p/\partial x + \rho_w g = F(r); \tag{3.8}$$

and on using (2.7) and (3.2), and integrating, it follows that

$$p(x,r) = xF(r) - \rho_0 g\{x + (\beta\tau/2a)x^2\} + G(r),$$

and so from (3.5) the unknown functions $F(r)$ and $G(r)$ must satisfy

$$F'(r) = -\beta\rho_0 m\tau E_s^2 a/r^3 \quad \text{and} \quad G'(r) = -\{\chi_0 - \beta\rho_0 m\theta(r)\}E_s^2 a^2/r^3.$$

Hence

$$F = \beta\rho_0 m\tau E_s^2 a/2r^2 + C, \tag{3.9}$$

where C is a pressure-drop constant, i.e. the dynamic pressure gradient when the electric field is zero. Finally the equation of motion (3.7) simplifies to

$$\nabla^2 u + (\beta g/\nu)\theta - \beta m\tau E_s^2 a/2\nu r^2 = \mu^{-1}C, \tag{3.10}$$

which together with equation (3.6) is to be solved subject to the boundary conditions

$$u(a) = u(b) = \theta(a) = \theta(b) = 0. \tag{3.11}$$

Equations (3.6), (3.10) and (3.11) can be expressed in non-dimensional form by introducing the new variables

$$R = r/a, \quad X = x/a, \quad U = au/k \quad \text{and} \quad \Theta = \theta/\tau. \tag{3.12}$$

These equations then become:

$$\frac{1}{R}\frac{d}{dR}\left(R\frac{d\Theta}{dR}\right) + U = 0, \tag{3.13}$$

$$\frac{1}{R}\frac{d}{dR}\left(R\frac{dU}{dR}\right) + A\Theta = \frac{\xi A}{2R^2} + \gamma, \tag{3.14}$$

and

$$U = \Theta = 0 \quad \text{at} \quad R = 1 \quad \text{and} \quad b/a. \tag{3.15}$$

Here $A = \beta g\tau a^3/k\nu$ is the Rayleigh number based on the inner tube radius and the temperature drop along the walls in a length equal to this tube radius; $\gamma = (a^3/k\nu\rho_0)\,C$ is a dimensionless pressure-drop constant; $\xi = mE_s^2/ga$ is a dimensionless group involved in the ratio of electrical buoyancy force to gravitational buoyancy force for the system.

For convenience let U and Θ be each divided into two parts as follows:

$$U = -\gamma U_c + \xi A U_e, \tag{3.16a}$$

and

$$\Theta = \gamma \Theta_c + \xi A \Theta_e. \tag{3.16b}$$

On substitution of (3.16a) and (3.16b) into equations (3.13) to (3.15), and equating to zero the coefficients of γ and ξ, the functions (U_c, Θ_c) and (U_e, Θ_e) are found to satisfy:

$$\left.\begin{array}{c}\dfrac{1}{R}\dfrac{d}{dR}\left(R\dfrac{d\Theta_c}{dR}\right) - U_c = 0, \quad \dfrac{1}{R}\dfrac{d}{dR}\left(R\dfrac{dU_c}{dR}\right) - A\Theta_c = -1, \\ \text{and} \quad U_c = \Theta_c = 0 \text{ at } R = 1 \quad \text{and} \quad b/a;\end{array}\right\} \tag{3.17}$$

$$\left.\begin{array}{c}\dfrac{1}{R}\dfrac{d}{dR}\left(R\dfrac{d\Theta_e}{dR}\right) - U_e = 0, \quad \dfrac{1}{R}\dfrac{d}{dR}\left(R\dfrac{dU_e}{dR}\right) + A\Theta_e = \dfrac{1}{2R^2}, \\ \text{and} \quad U_e = \Theta_e = 0 \text{ at } R = 1 \quad \text{and} \quad b/a.\end{array}\right\} \tag{3.18}$$

Physically $-\gamma U_c$ and $\gamma \Theta_c$ are the dimensionless velocity and temperature components due to the applied pressure gradient and the gravitational buoyancy force; $\xi A U_e$ and $\xi A \Theta_e$ are the dimensionless velocity and temperature components induced by the electrical buoyancy force.

If Θ_c is eliminated from (3.17), the result is

$$\left(\frac{d^2}{dR^2} + \frac{1}{R}\frac{d}{dR}\right)^2 U_c = 0, \tag{3.19a}$$

subject to the boundary conditions

$$U_c(1) = U_c(b/a) = 0, \quad \left(\frac{d^2}{dR^2} + \frac{1}{R}\frac{d}{dR}\right) U_c = -1 \quad \text{at} \quad R = 1 \text{ and } b/a, \tag{3.19b}$$

respectively. The solution of (3.19a) and (3.19b) can be expressed in terms of the zero-order Bessel functions $J_0(A^{\frac{1}{4}}R)$ and $Y_0(A^{\frac{1}{4}}R)$ and the modified Bessel functions of zero order $I_0(A^{\frac{1}{4}}R)$ and $K_0(A^{\frac{1}{4}}R)$. The solution is

$$U_c = c_1 J_0(A^{\frac{1}{4}}R) + c_2 Y_0(A^{\frac{1}{4}}R) + c_3 I_0(A^{\frac{1}{4}}R) + c_4 K_0(A^{\frac{1}{4}}R), \tag{3.19c}$$

where the constants of integration c_i are:

$$\left.\begin{array}{ll}2A^{\frac{1}{4}}\Delta_1 c_1 = Y_0(bA^{\frac{1}{4}}/a) - Y_0(A^{\frac{1}{4}}), & 2A^{\frac{1}{4}}\Delta_1 c_2 = J_0(A^{\frac{1}{4}}) - J_0(bA^{\frac{1}{4}}/a), \\ 2A^{\frac{1}{4}}\Delta_2 c_3 = K_0(A^{\frac{1}{4}}) - K_0(bA^{\frac{1}{4}}/a), & 2A^{\frac{1}{4}}\Delta_2 c_4 = I_0(bA^{\frac{1}{4}}/a) - I_0(A^{\frac{1}{4}}).\end{array}\right\} \tag{3.19d}$$

Here

$$\Delta_1 = J_0(A^{\frac{1}{4}})\, Y_0(bA^{\frac{1}{4}}/a) - J_0(bA^{\frac{1}{4}}/a)\, Y_0(A^{\frac{1}{4}}), \tag{3.20}$$

and

$$\Delta_2 = I_0(A^{\frac{1}{4}})\, K_0(bA^{\frac{1}{4}}/a) - I_0(bA^{\frac{1}{4}}/a)\, K_0(A^{\frac{1}{4}}). \tag{3.21}$$

Moreover, if Θ_e is eliminated from (3.18), the result is

$$\left(\frac{d^2}{dR^2} + \frac{1}{R}\frac{d}{dR}\right)^2 U_e = \frac{2}{R^4}, \tag{3.22a}$$

subject to the boundary conditions

$$U_e(1) = U_e(b/a) = 0, \quad \left(\frac{d^2}{dR^2} + \frac{1}{R}\frac{d}{dR}\right) U_e = \frac{1}{2R^2} \quad \text{at} \quad R = 1 \quad \text{and} \quad b/a, \tag{3.22b}$$

respectively. Introducing the functions

$$\left.\begin{aligned} B_0(A^{\frac{1}{4}}R) &= J_0(A^{\frac{1}{4}}R)\int_1^R \frac{1}{R}Y_0(A^{\frac{1}{4}}R)\,dR - Y_0(A^{\frac{1}{4}}R)\int_1^R \frac{1}{R}J_0(A^{\frac{1}{4}}R)\,dR, \\ \text{and}\quad B_1(A^{\frac{1}{4}}R) &= I_0(A^{\frac{1}{4}}R)\int_1^R \frac{1}{R}K_0(A^{\frac{1}{4}}R)\,dR - K_0(A^{\frac{1}{4}}R)\int_1^R \frac{1}{R}I_0(A^{\frac{1}{4}}R)\,dR, \end{aligned}\right\} \quad (3.22c)$$

the solution of $(3.22a)$ and $(3.22b)$ can be expressed in the form:

$$\begin{aligned} U_e = d_1 J_0(A^{\frac{1}{4}}R) + d_2 Y_0(A^{\frac{1}{4}}R) + d_3 I_0(A^{\frac{1}{4}}R) \\ + d_4 K_0(A^{\frac{1}{4}}R) - \tfrac{1}{8}\pi B_0(A^{\frac{1}{4}}R) + \tfrac{1}{4}B_1(A^{\frac{1}{4}}R). \end{aligned} \quad (3.22d)$$

The constants of integration d_i were obtained as follows:

$$\left.\begin{aligned} 8\Delta_1 d_1 &= -\pi Y_0(A^{\frac{1}{4}})\,B_0(bA^{\frac{1}{4}}/a), & 8\Delta_1 d_2 &= \pi J_0(A^{\frac{1}{4}})\,B_0(bA^{\frac{1}{4}}/a), \\ 4\Delta_2 d_3 &= K_0(A^{\frac{1}{4}})\,B_1(bA^{\frac{1}{4}}/a), & 4\Delta_2 d_4 &= -I_0(A^{\frac{1}{4}})\,B_1(bA^{\frac{1}{4}}/a). \end{aligned}\right\} \quad (3.22e)$$

The integrals occurring in equations $(3.22c)$ cannot be expressed in terms of Bessel functions but can be evaluated numerically by quadratures.† Expressions for Θ_c and Θ_e are not given as these are readily obtained using equations (3.17) and (3.18) and expressions $(3.19c)$ and $(3.22d)$.

Examination of (3.20) shows that there are critical values of A for which $\Delta_1 = 0$.‡ Then the constants of integration c_1, c_2, d_1, d_2, occurring in expressions $(3.19d)$ and $(3.22e)$, are infinite thus giving infinite velocity and temperature distributions. For example, if $b/a = 2$, the critical Rayleigh numbers are $A^{\frac{1}{4}} = 3{\cdot}1230, 6{\cdot}2734$, etc. Non-dimensional velocity functions U_c and U_e are shown in figure 1, and the temperature functions Θ_c and Θ_e in figure 2, for $A^{\frac{1}{4}} = 1$, 2 and 3, taking $b/a = 2$. It can be seen that $|U_c|$, $|U_e|$, $|\Theta_c|$ and $|\Theta_e|$ increase at first slowly with increasing A and then increase extremely rapidly as the critical value $A^{\frac{1}{4}} = 3{\cdot}1230$ is approached. In the terminology of Morton (1960) the solution is then said to 'run-away'. The explanation given by Morton is that as the temperature gradient along the walls is increased, greater buoyancy forces are produced leading to higher gas velocities, and so to a further increase in the buoyancy force. In the range $3{\cdot}1230 < A < 6{\cdot}2734$ the velocity and temperature functions were found to have a different character. In fact U_c, U_e, Θ_c and Θ_e were then found to alternate once in sign indicating an up-flow and down-flow near the inner and outer walls, respectively. Furthermore, as pointed out by Morton, the more complicated flows that theoretically exist above $A^{\frac{1}{4}} = 3{\cdot}1230$ would probably not occur in practice, as they would require improbable end conditions. Moreover, at or near $A^{\frac{1}{4}} = 3{\cdot}1230$ the laminar similarity solution discussed in this section, would probably be in error since in practice the flow may become unstable and then turbulent.

From figures 1 and 2 the dimensionless velocity and temperature profiles may be obtained for specified values of the parameters γ and ξ (see expressions $(3.16a)$

† The integral $Ji_0(z) = \int_z^\infty \frac{1}{z}\,J_0(z)\,dz$ has been evaluated by Lowan, Blanch & Ambramowitz (1943).

‡ The roots of the transcendental equation $\Delta_1 = 0$, for various values of b/a, have been tabulated by Carslaw & Jaeger (1947, p. 379).

and (3.16*b*)). Physically $\gamma > 0$ corresponds to a heated down-flow, whilst $\gamma < 0$ corresponds to a cooled up-flow; these flows are due to the applied pressure gradient and the gravitational buoyancy force. However, $\xi > 0$ corresponds to a heated down-flow induced by the electrical buoyancy force. Thus the gas velocity in a cooled up-flow between vertical concentric cylinders will be reduced in magnitude if an electrostatic potential difference exists between the cylinders.

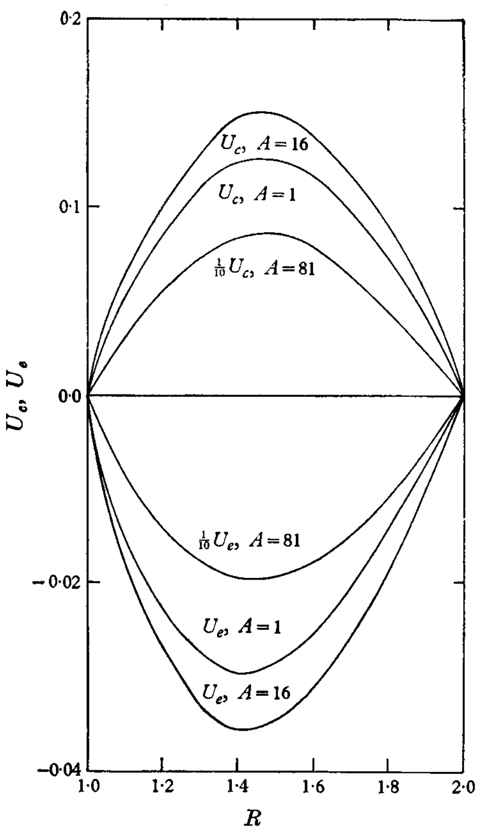

FIGURE 1. Non-dimensional velocity functions U_c and U_e for $A = 1, 16, 81$. Actual non-dimensional velocity $U = -\gamma U_c + \xi A U_e$.

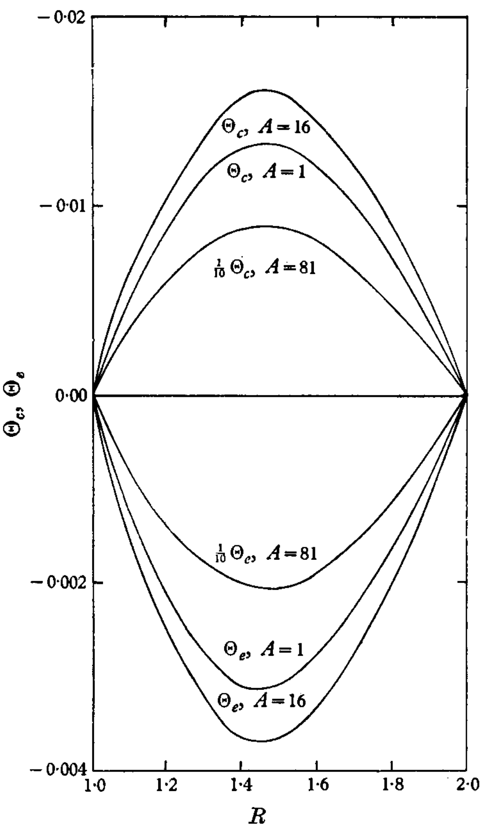

FIGURE 2. Non-dimensional temperature functions Θ_c and Θ_e for $A = 1, 16, 81$. Actual non-dimensional temperature $\Theta = \gamma\Theta_c + \xi A \Theta_e$.

It is perhaps instructive to establish an approximate condition for the electrical buoyancy effect to be comparable in magnitude with the combined effect of the pressure gradient and the gravitational buoyancy effect. Consider the relative magnitude of the terms on the right-hand side of equation (3.14). If, say,

$$\xi A \int_1^{b/a} \frac{1}{2R^2} R\,dR = O\left(\left|\gamma \int_1^{b/a} R\,dR\right|\right),$$

i.e.

$$\xi = O\left(\left|\frac{\gamma(b^2-a^2)}{Aa^2 \log (b/a)}\right|\right), \tag{3.23}$$

the electrical and combined pressure-gradient and gravitational effects will be of the same order. Taking $b/a = 2$, the condition (3.23) gives $\xi = O(4\cdot 4\,\gamma/A)$, a result verified by the exact calculations shown in figures 1 and 2, where it is seen that the ratios $|U_c|:|U_e|$ and $|\Theta_c|:|\Theta_e|$, at any R, are both approximately in the ratio 5 : 1. Having established the validity of (3.23) it follows that the gas velocity in a cooled up-flow ($\gamma < 0$) will be virtually reduced to zero if an electric field of magnitude

$$E_s = O\left\{\left|\frac{g\gamma(b^2-a^2)}{mAa\log(b/a)}\right|^{\frac{1}{2}}\right\} \quad \text{e.s.u.}$$

is applied at the inner cylinder.

The heat transfer coefficients at the inner cylinder $r = a$ and the outer cylinder $r = b$ can be expressed in terms of the Nusselt numbers

$$Nu_0 = \frac{a}{\tau}\left(\frac{d\theta}{dr}\right)_{r=a} \quad \text{and} \quad Nu_1 = \frac{a}{\tau}\left(\frac{d\theta}{dr}\right)_{r=b}, \tag{3.24}$$

respectively. In terms of dimensionless quantities

$$Nu_0 = \left(\frac{d\Theta}{dR}\right)_{R=1} = \left(\gamma\frac{d\Theta_c}{dR} + \xi A\frac{d\Theta_e}{dR}\right)_{R=1} = \gamma n_c^{(0)} + \xi A n_e^{(0)}, \tag{3.25}$$

and

$$Nu_1 = \left(\frac{d\Theta}{dR}\right)_{R=b/a} = \left(\gamma\frac{d\Theta_c}{dR} + \xi A\frac{d\Theta_e}{dR}\right)_{R=b/a} = \gamma n_c^{(1)} + \xi A n_e^{(1)}. \tag{3.26}$$

Here

$$\left.\begin{aligned} A^{\frac{1}{4}}n_c^{(0)} &= [c_1J_1(A^{\frac{1}{4}}) + c_2Y_1(A^{\frac{1}{4}}) + c_3I_1(A^{\frac{1}{4}}) + c_4K_1(A^{\frac{1}{4}})],\\ A^{\frac{1}{4}}n_c^{(1)} &= [c_1J_1(bA^{\frac{1}{4}}/a) + c_2Y_1(bA^{\frac{1}{4}}/a) + c_3I_1(bA^{\frac{1}{4}}/a) + c_4K_1(bA^{\frac{1}{4}}/a)];\end{aligned}\right\} \tag{3.25a}$$

$$\left.\begin{aligned} A^{\frac{1}{4}}n_e^{(0)} &= -[d_1J_1(A^{\frac{1}{4}}) + d_2Y_1(A^{\frac{1}{4}}) + d_3I_1(A^{\frac{1}{4}}) + d_4K_1(A^{\frac{1}{4}})],\\ \text{and}\quad A^{\frac{1}{4}}n_e^{(1)} &= -\Bigg[d_1J_1(bA^{\frac{1}{4}}/a) + d_2Y_1(bA^{\frac{1}{4}}/a) + d_3I_1(bA^{\frac{1}{4}}/a)\\ &\quad + d_4K_1(bA^{\frac{1}{4}}/a) + \frac{1}{4}\Bigg\{I_1(bA^{\frac{1}{4}}/a)\int_1^{b/a}\frac{1}{R}K_0(A^{\frac{1}{4}}R)\,dR\\ &\quad - K_1(bA^{\frac{1}{4}}/a)\int_1^{b/a}\frac{1}{R}I_0(A^{\frac{1}{4}}R)\,dR\Bigg\}\\ &\quad - \frac{\pi}{8}\Bigg\{J_1(bA^{\frac{1}{4}}/a)\int_1^{b/a}\frac{1}{R}Y_0(A^{\frac{1}{4}}R)\,dR - Y_1(bA^{\frac{1}{4}}/a)\int_1^{b/a}\frac{1}{R}J_0(A^{\frac{1}{4}}R)\,dR\Bigg\}\Bigg].\end{aligned}\right\} \tag{3.26a}$$

In expressions (3.25a) and (3.26a) J_1 and Y_1 are Bessel functions of the first order and I_1 and K_1 are modified Bessel functions of the first order. The non-dimensional quantities $n_c^{(0)}$, $n_e^{(0)}$, $n_c^{(1)}$ and $n_e^{(1)}$ are shown in figure 3 for $b/a = 2$ and

$$0 < A^{\frac{1}{4}} < 3\cdot 1230,$$

the first critical Rayleigh number; there is little variation in these quantities for small Rayleigh numbers, but as the first critical Rayleigh number is approached there is a large variation.

The rate of volume flow through the annulus is

$$Q = 2\pi\int_a^b ur\,dr = 2\pi ak\int_1^{b/a}(-\gamma U_c + \xi AU_e)\,R\,dR. \tag{3.27}$$

From (3.17), (3.18), (3.25) and (3.26) it follows that

$$\int_1^{b/a} U_c R\,dR = \frac{b}{a}\left(\frac{d\Theta_c}{dR}\right)_{R=b/a} - \left(\frac{d\Theta_c}{dR}\right)_{R=1} = \frac{b}{a}n_c^{(1)} - n_c^{(0)},$$

and

$$\int_1^{b/a} U_e R\,dR = \frac{b}{a}\left(\frac{d\Theta_e}{dR}\right)_{R=b/a} - \left(\frac{d\Theta_e}{dR}\right)_{R=1} = \frac{b}{a}n_e^{(1)} - n_e^{(0)}.$$

Hence the rate of volume flow through the annulus can be simply expressed in the form

$$Q = 2\pi ak(-\gamma q_c + \xi A q_e), \tag{3.28}$$

where $q_c = (b/a)\,n_c^{(1)} - n_c^{(0)}$ and $q_e = (b/a)\,n_e^{(1)} - n_e^{(0)}$.

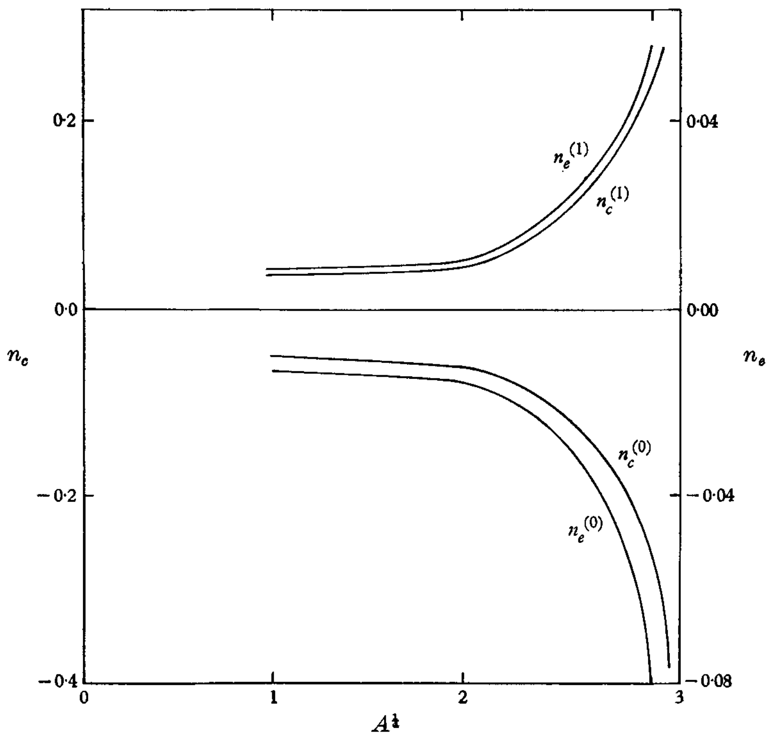

FIGURE 3. Quantities $n_c^{(0)}$, $n_c^{(1)}$, $n_e^{(0)}$ and $n_e^{(1)}$ shown as functions of $A^{\frac{1}{4}}$ up to the first critica Rayleigh number. At the inner wall the Nusselt number $Nu_0 = \gamma n_c^{(0)} + \xi A n_e^{(0)}$, and at the outer wall the Nusselt number $Nu_1 = \gamma n_c^{(1)} + \xi A n_e^{(1)}$. The volume rate of flow through the annulus is $Q = 2\pi ak(\gamma q_c + \xi A q_e)$, where $q_c = (b/a)\,n_c^{(1)} - n_c^{(0)}$ and $q_e = (b/a)\,n_e^{(1)} - n_e^{(0)}$.

The quantities q_c and q_e for $b/a = 2$ and $0 < A^{\frac{1}{4}} < 3{\cdot}1230$ are readily obtained from the results shown in figure 3.

Consider, as an illustration of the above theory, the convective motion of ammonia gas between concentric cylinders of radius $a = \frac{1}{2}$ cm and $b = 1$ cm, and for which $\gamma = \pm 250$ and $A = 16$. First, since $A = \beta g \tau a^3/k\nu$ and taking $T_0 = 25$ °C, it follows from table 1 that there must be a wall temperature decrease of 0·8 °C/cm; on using (3.12), (3.16a) and (3.16b) it follows from figures 1 and 2 that at $r = 0{\cdot}75$ cm the actual velocity $u_c = \mp 11$ cm/sec and the actual temperature $\theta_c = \mp 3$ °C. If now the potential difference between the inner and outer cylinder is 7·3 kV then $mE_s^2/ga = 50$, and so at $r = 0{\cdot}75$ cm the electrical induced velocity and temperature are $u_e = -8$ cm/sec and $\theta_e = -1$ °C. Furthermore, under these conditions $\gamma n_c^{(0)} = \mp 15$, $\xi A n_e^{(0)} = -13$ at the inner cylinder; also $\gamma n_c^{(1)} = \pm 10$ and $\xi A n_e^{(1)} = +6$ at the outer cylinder. Hence if

$$dT_w/dx = -0{\cdot}8\ ^\circ\text{C/cm}, \quad a = \tfrac{1}{2}\,\text{cm}, \quad b = 1\,\text{cm}$$

and the potential difference is 7·3 kV, then (i) the actual velocity $u = -19$ cm/sec and temperature $\theta = -4$ °C at $r = 0{\cdot}75$ cm, provided $\gamma = +250$; and (ii) $u = 3$ cm/sec, $\theta = +2$ °C at $r = 0{\cdot}75$ cm, if $\gamma = -250$. Moreover, the Nusselt

$N = 6{\cdot}0228 \times 10^{23}$ per g mol	$\rho = 7{\cdot}7 \times 10^{-3}$ g/cm³
$M = 17$	
$\sigma = 63$ cm³/g mol	$\nu = 0{\cdot}14$ cm²/sec
$\mu = 1{\cdot}44 \times 10^{-18}$ dyn cm	$k = 0{\cdot}145$ cm²/sec
$k_B = 1{\cdot}380\ 10^{-16}$ erg/°C	$m = 4{\cdot}94$ cm³/mol

TABLE 1. Physical constants for ammonia gas (Loeb 1927; Kaye & Laby 1928; Childs 1946).

number at the inner wall due to a pressure gradient $\gamma = -250$ is decreased by 85 % and at the outer wall there is an increase of 60 % if there exists a 7·3 kV potential difference. The signs of the above percentages are changed if $\gamma = +250$.

It should be noted that there is a limit to the magnitude of the potential difference between the cylinders. As far as the author is aware information is not available for ammonia but this should not be too different from the data for air for which the peak sparking voltage is approximately in the range 15 to 20 kV for a $\frac{1}{2}$ cm spark gap (see Kaye & Laby 1928).

Finally the above analysis has been developed for an unstable thermal stratification of the gas. Stable thermal stratification is obtained if the wall temperature increases with height and a corresponding analysis for a paraelectric gas with an imposed radial electric field could be developed but will not be considered in the present paper (see Morton 1960).

4. Convection due to a heated charged sphere in a paraelectric gas

Consider a sphere of radius a maintained at a constant temperature T_1, where $T_1 > T_0$ the ambient temperature of the surrounding paraelectric gas; the sphere is charged to potential V_a V. The system will be referred to spherical polar co-ordinates (r, θ, ϕ), θ being measured from the lower stagnation point. From equations (2.16) it follows that the electric field components are:

$$E_r = E_s a^2/r^2, \quad (E_\theta = E_\phi = 0), \tag{4.1}$$

where

$$E_s = V_a/300a \text{ e.s.u.}$$

Due to symmetry the system will be independent of ϕ. The equations to be solved are given in §2. On elimination of the pressure from equation (2.2) there results the equation:

$$-\rho\,\text{curl}\,(\mathbf{v} \times \text{curl}\,\mathbf{v}) = \text{curl}\,\mathbf{F} - \rho\nu\,\text{curl}\,\text{curl}\,\text{curl}\,\mathbf{v}. \tag{4.2}$$

The equation of continuity is satisfied identically by the components

$$\mathbf{v} = \left(-\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta},\quad \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r},\quad 0\right), \tag{4.3}$$

where ψ is the stream function. On substitution of (4.3) into (4.2) and (2.3) and using (4.1) and (2.14), there results a pair of coupled non-linear partial differential equations in ψ and T, to be solved subject to the boundary conditions

$$\mathbf{v} = 0,\quad T = T_1 \quad \text{on} \quad r = a, \quad \text{and} \quad \mathbf{v} \to 0,\quad T \to T_0 \quad \text{as} \quad r \to \infty.$$

A complete solution of these equations would be a formidable task and herein is given an approximate solution of the corresponding boundary-layer-type equations, which are now derived.

Let units of length, electric field strength, temperature difference and velocity be a, E_s, $T_1 - T_0$ and ν/a respectively. Dimensional analysis of the ψ and T equations leads to the introduction of the three dimensionless parameters: $G = \beta g a^3(T_1 - T_0)/\nu^2$, the Grashof number; $P = \nu/k$, the Prandtl number; and $\xi = mE_s^2/ga$, already introduced in § 3. Physically speaking, the Grashof number represents the ratio of the gravitational buoyancy force to the viscous forces. The laminar convection flows of interest here are those associated with moderately large Grashof numbers, i.e. $G \geqslant 10^4$. This restriction implies that all viscous and thermal effects are confined to a thin boundary layer next to the heated charged sphere. Following the usual boundary-layer theory approach for obtaining an asymptotic solution for large G of the ψ and T equations, it was found convenient to introduce the new variables:

$$R = r/a = 1 + G^{-\frac{1}{4}}y,\quad x = \theta,\quad \Phi = \psi/\nu a G^{\frac{1}{4}},\quad \Theta = (T - T_0)/(T_1 - T_0). \tag{4.4}$$

The equation of motion (4.2) is now expressed in terms of these variables. Thus on using (4.3) it follows that:

$$\rho\nu\,\mathrm{curl\,curl\,curl}\,\mathbf{v} = \left\{0,\,0,\,\frac{\rho\nu^2}{a^4}G^{\frac{5}{4}}\left[-\frac{1}{R\sin x}\frac{\partial^4\Phi}{\partial y^4} + O(G^{-\frac{1}{2}})\right]\right\}, \tag{4.5a}$$

and

$$\rho\,\mathrm{curl}\,(\mathbf{v}\times\mathrm{curl}\,\mathbf{v}) = \left\{0,0,\frac{\rho\nu^2}{a^4}G^{\frac{5}{4}}\left[\frac{1}{R^3\sin^2 x}\left(\frac{\partial\Phi}{\partial x}\frac{\partial^3\Phi}{\partial y^3} - \frac{\partial\Phi}{\partial y}\frac{\partial^3\Phi}{\partial y^2\,\partial x}\right)\right.\right.$$
$$\left.\left. + \frac{2\cos x}{R^3\sin^3 x}\frac{\partial\Phi}{\partial y}\frac{\partial^2\Phi}{\partial y^2} - \frac{2G^{-\frac{1}{4}}}{R^3\sin^2 x}\frac{\partial\Phi}{\partial x}\frac{\partial^2\Phi}{\partial y^2} + O(G^{-\frac{1}{2}})\right]\right\}; \tag{4.5b}$$

furthermore, on using (2.14) and (4.1), there results the expression

$$\mathrm{curl}\,\mathbf{F} = \left\{0,0,\frac{\rho\nu^2}{a^4}G^{\frac{5}{4}}\left[\sin x\frac{\partial\Theta}{\partial y} + \frac{G^{-\frac{1}{4}}}{R}\left(\cos x - \frac{2\xi}{R^5}\right)\frac{\partial\Theta}{\partial x}\right]\right\}. \tag{4.5c}$$

On substitution of (4.5a), (4.5b) and (4.5c) into (4.2) there results the equation

$$\frac{1}{R}\frac{\partial^4\Phi}{\partial y^4} + \sin x\frac{\partial\Theta}{\partial y} + \frac{1}{R^3}\left[\frac{1}{\sin^2 x}\left(\frac{\partial\Phi}{\partial x}\frac{\partial^3\Phi}{\partial y^3} - \frac{\partial\Phi}{\partial y}\frac{\partial^3\Phi}{\partial x\,\partial y^2}\right) - \frac{2\cos x}{\sin^3 x}\frac{\partial\Phi}{\partial y}\frac{\partial^2\Phi}{\partial y^2}\right]$$
$$= G^{-\frac{1}{4}}\frac{1}{R}\left[\frac{1}{R^3\sin^2 x}\frac{\partial\Phi}{\partial x}\frac{\partial^2\Phi}{\partial y^2} - \left(\cos x - \frac{2\xi}{R^5}\right)\frac{\partial\Theta}{\partial x}\right] + O(G^{-\frac{1}{2}}). \tag{4.6}$$

Further simplification of this equation is possible on introducing the dimensionless velocity components:

$$U = \frac{1}{\sin x}\frac{\partial \Phi}{\partial y} \quad \text{and} \quad W = -\frac{1}{\sin x}\frac{\partial \Phi}{\partial x}, \tag{4.7}$$

where U and W satisfy the equation of continuity

$$\partial/\partial x\,(U \sin x) + \partial/\partial y\,(W \sin x) = 0. \tag{4.8}$$

Then equation (4.6) is reduced to

$$\frac{1}{R}\frac{\partial^3 U}{\partial y^3} + \sin x\frac{\partial \Theta}{\partial y} - \frac{1}{R^3}\frac{\partial}{\partial y}\left(U\frac{\partial U}{\partial x} + W\frac{\partial U}{\partial y}\right)$$
$$= G^{-\frac{1}{4}}\frac{1}{R}\left[\frac{2}{R^3}W\frac{\partial U}{\partial y} + \left(\cos x - \frac{2\xi}{R^5}\right)\frac{\partial \Theta}{\partial x}\right] + O(G^{-\frac{1}{2}}). \tag{4.9}$$

Similarly, on using (4.4) and (4.7), the temperature equation (2.3) becomes

$$R^2\frac{\partial^2 \Theta}{\partial y^2} - P\left(U\frac{\partial \Theta}{\partial x} + W\frac{\partial \Theta}{\partial y}\right) = -2G^{-\frac{1}{4}}R\frac{\partial \Theta}{\partial y}. \tag{4.10}$$

Equations (4.8), (4.9) and (4.10) are to be solved subject to the boundary conditions:

$$U = W = 0, \quad \Theta = 1 \quad \text{at} \quad y = 0, \quad \text{and} \quad U \to 0, \quad \Theta \to 0 \quad \text{as} \quad y \to \infty. \tag{4.11}$$

There are now several points to be discussed concerning equations (4.8) to (4.11). It should be noted that when G is very large such that terms of $O(G^{-\frac{1}{4}})$ may be neglected, then $R \simeq 1$ and the terms on the right-hand sides of (4.9) and (4.10) vanish. There results, if $\xi = 0$, the conventional boundary-layer equations for the free convection from a heated sphere:†

$$\left.\begin{aligned} &\frac{\partial^2 U}{\partial y^2} + \Theta \sin x = U\frac{\partial U}{\partial x} + W\frac{\partial U}{\partial y}, \\ &\frac{\partial}{\partial x}(U \sin x) + \frac{\partial}{\partial y}(W \sin x) = 0, \\ \text{and} \qquad &\frac{\partial^2 \Theta}{\partial y^2} - P\left(U\frac{\partial \Theta}{\partial x} + W\frac{\partial \Theta}{\partial y}\right) = 0. \end{aligned}\right\} \tag{4.12}$$

These equations assume that the pressure across the boundary layer is constant. However, when G is moderately large (i.e. $10^4 \leqslant G \leqslant 10^6$), terms of $O(G^{-\frac{1}{4}})$ in equations (4.9) and (4.10) should not be neglected. In fact those terms of $O(G^{-\frac{1}{4}})$ on the right-hand side of equation (4.9) which involve the temperature give rise to a pressure gradient across the boundary layer; these terms are due to the radial component of the gravitational buoyancy force and to the electrical buoyancy force. Thus, to examine the effect of a radial electric field on the free-convection from a heated sphere, terms of $O(G^{-\frac{1}{4}})$ in equations (4.9) to (4.11) must be

† Merk & Prins (1954) have obtained approximate solutions to these equations, for a wide range of Prandtl numbers, using a method developed by Squire (1938).

retained. Moreover, to the same approximation, the variable R cannot be replaced by unity. In a similar fashion the pressure gradient across the boundary layer has been taken into account by Levy (1955) in a theoretical investigation of the free convection from a heated horizontal cylinder.

A modified momentum equation is now obtained from (4.9) as follows. Since U and $\Theta \to 0$ as $y \to \infty$ it is reasonable to assume in addition that $\partial U/\partial y$, $R\,\partial U/\partial y$, $R^2\,\partial^2 U/\partial y^2$ and $R^2\Theta \to 0$ as $y \to \infty$; these assumptions are probably equivalent to the assumption that U and Θ decay at least exponentially to zero as $y \to \infty$. On multiplying both sides of equation (4.9) by $R^2\,dy$ and integrating with respect to y from y to ∞, and using the equation of continuity (4.8), there results:

$$\begin{aligned} R\frac{\partial^2 U}{\partial y^2} + R^2\Theta\sin x - \frac{1}{R}\left(U\frac{\partial U}{\partial x} + W\frac{\partial U}{\partial y}\right) \\ = G^{-\frac{1}{4}}\Bigg[\frac{\partial U}{\partial y} - 2\sin x\int_y^\infty R\Theta\,dy - \frac{WU}{R^2} \\ + \cot x\int_y^\infty\left(\frac{U}{R}\right)^2 dy + \int_y^\infty\left(R\cos x - \frac{2\xi}{R^4}\right)\frac{\partial\Theta}{\partial x}\,dy\Bigg] + O(G^{-\frac{1}{2}}). \end{aligned} \tag{4.13}$$

This equation, together with the thermal equation (4.10), may now be solved approximately using the method of Squire (1938). The actual range of integration $y = 0$ to ∞ is replaced by the effective range $y = 0$ to δ, where δ is the boundary-layer thickness. In doing so it is assumed that the hydrodynamic and thermal boundary-layer thicknesses are equal. This assumption is exact if $P = 1$ and only approximately true when $P \sim 1$. The dimensionless velocity and thermal profiles are taken to satisfy

$$\left.\begin{aligned} U = 0,\quad \Theta = 1 \quad \text{at} \quad y = 0, \\ U = 0,\quad \Theta = 0,\quad \frac{\partial U}{\partial y} = \frac{\partial\Theta}{\partial y} = 0 \quad \text{at} \quad y = \delta. \end{aligned}\right\} \tag{4.14}$$

These conditions are satisfied by

$$U = F(x)\frac{y}{\delta}\left(1 - \frac{y}{\delta}\right)^2 \quad \text{and} \quad \Theta = \left(1 - \frac{y}{\delta}\right)^2. \tag{4.15}$$

Modified momentum and thermal integral equations are obtained on integrating equations (4.10) and (4.13) across the boundary layer from $y = 0$ to $y = \delta$ and using (4.8) and (4.14). These are:

$$\begin{aligned} -\left(\frac{\partial U}{\partial y}\right)_0 + \sin x\int_0^\delta R^2\Theta\,dy - \left(\frac{\partial}{\partial x} + \cot x\right)\int_0^\delta \frac{U^2}{R}\,dy \\ = G^{-\frac{1}{4}}\Bigg[-2\sin x\int_0^\delta\left(\int_y^\delta R\Theta\,dy\right)dy + \cot x\int_0^\delta\left(\int_0^\delta\left(\frac{U}{R}\right)^2 dy\right)dy \\ + \int_0^\delta\left(\int_y^\delta\left(R\cos x - \frac{2\xi}{R^4}\right)\frac{\partial\Theta}{\partial x}\,dy\right)dy\Bigg] + O(G^{-\frac{1}{2}}), \end{aligned} \tag{4.16}$$

and
$$-\left(\frac{\partial\Theta}{\partial y}\right)_0 = P\left(\frac{\partial}{\partial x} + \cot x\right)\int_0^\delta U\Theta\,dy. \tag{4.17}$$

On substitution of the assumed profiles (4.15) into (4.16) and (4.17) there results, after some algebra, the following equations for determining the unknown functions $F(x)$ and $\delta(x)$:

$$F = \frac{60}{P}\frac{1}{\delta \sin x}\int_0^x \frac{\sin x}{\delta}\,dx, \tag{4.18}$$

$$\begin{aligned} &F^2\delta\, d\delta/dx + 35\delta^2 \sin x + F^2\delta^2 \cot x - 15(7+8/P)\,F \\ &\quad + G^{-\frac{1}{4}}\delta\,[(45/P)\,F + 35\delta^2 \sin x - \tfrac{3}{4}F^2\delta^2 \cot x - \tfrac{35}{2}(\cos x - 2\xi)\,\delta\, d\delta/dx] \\ &\quad + O(G^{-\frac{1}{2}}) = 0, \end{aligned} \tag{4.19}$$

subject to the boundary condition

$$d\delta/dx = 0 \quad \text{at} \quad x = 0. \tag{4.20}$$

These equations have been solved numerically, using a step-by-step finite difference method, for the following cases:

$$G^{\frac{1}{4}} = 20,\ 25,\ 30 \text{ and } 40,$$
$$\xi = mE_s^2/ga = 0,\ 1,\ 5 \text{ and } 10,$$

and taking $P = 0{\cdot}98$ for ammonia gas.

The mean value of the Nusselt number is defined as

$$\overline{Nu} = \frac{1}{2}\int_0^\pi Nu \sin\theta\, d\theta = \frac{1}{2}\int_0^\pi \frac{a}{T_1 - T_0}\left(\frac{\partial T}{\partial r}\right)_{r=a} \sin\theta\, d\theta; \tag{4.21}$$

in terms of the dimensionless moduli we obtain

$$\frac{\overline{Nu}}{G^{\frac{1}{4}}} = \frac{1}{2}\int_0^\pi \left(\frac{\partial \Theta}{\partial y}\right)_{y=0} \sin x\, dx = \int_0^\pi \frac{\sin x}{\delta}\,dx. \tag{4.22}$$

Results for $\overline{Nu}/G^{\frac{1}{4}}$ are shown in figure 4 for the above values of $G^{\frac{1}{4}}$ and ξ. For example if the sphere has a diameter of 5 cm then $\xi = 0$, 1, 5 and 10 corresponds to the sphere being maintained at the potentials 0, 16·17, 37·35 and 52·84 kV, respectively (see table 1).

The influence of a radial electric field on the convective motion due to a heated sphere in a paraelectric gas may be summarized as follows.

(i) As the applied electric field increases the electrical buoyancy force increases causing a decrease in the boundary-layer thickness and so an increase in the heat transfer at the surface. Thus if $a = 2{\cdot}5$ cm, $T_0 = 20\,^\circ$C and $T_1 - T_0 = 60\,^\circ$C then for ammonia $G^{\frac{1}{4}} = 20$; as seen from figure 4 the electrostatic potentials 16·71, 37·35 and 52·84 kV will then produce in $\overline{Nu}/G^{\frac{1}{4}}$ increases of $3\frac{1}{2}$, 16 and 28 %, respectively.

(ii) The presence of the electric field will probably delay separation near the upper pole of the sphere. However, information cannot be gained on this topic from the above approximate method of solution of the basic equations of motion.

Now with regard to the method of solution adopted it is evident that the results obtained on the free convection ($\xi = 0$) from a heated sphere in ammonia gas are more accurate than those calculated from the conventional laminar boundary-layer equations (4.12). For confirmation with experimental results a similar calculation was carried out for the free convection from a heated sphere in air. Recent measurements by Bromham & Mayhew (1962) on the laminar free convection from a heated sphere, having a diameter equal to 4 inches, gave the results $\overline{Nu} = 0{\cdot}378\,G^{\frac{1}{4}}$, for the range $4\times 10^5 \leqslant G \leqslant 10^6$. Taking $P = 0{\cdot}7$ and $G = 8{\cdot}1\times 10^5$, then Squire's procedure applied to the modified boundary-layer equations (4.10) and (4.13) gave $\overline{Nu} = 0{\cdot}369\,G^{\frac{1}{4}}$, whilst when applied to the conventional boundary-layer equations (4.12) Merk & Prins (1954) obtained $\overline{Nu} = 0{\cdot}365\,G^{\frac{1}{4}}$. Obviously the difference between theory and experiment, in this range of Grashof number, is not explained by accounting for such terms as

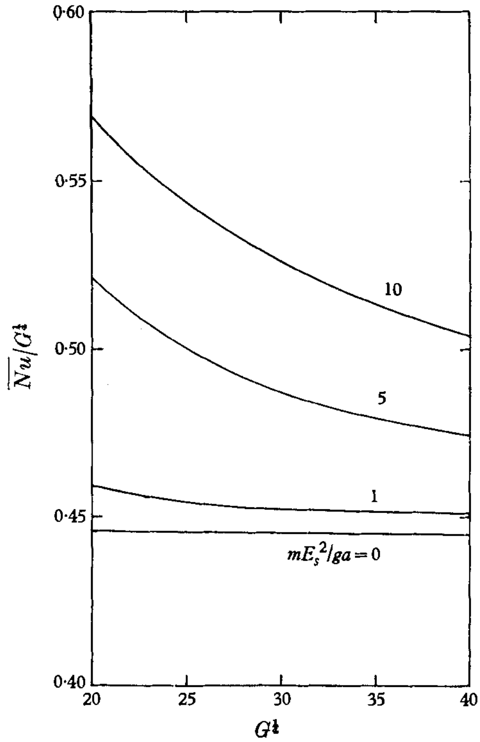

FIGURE 4. Mean Nusselt number $\overline{Nu}$ for a charged sphere in ammonia gas for various $\xi = mE_s^2/ga$.

the pressure gradient across the boundary layer. The discrepancy is probably due to the basic assumptions of § 2 together with an inadequate treatment of the plume at the upper pole of the sphere. However, if the gas is paraelectric and a radial field is imposed then there will be a substantial modification in the pressure across the boundary layer due to the induced electrical buoyancy force. Thus the method of solution adopted should be sufficient to predict general trends and the correct order of magnitude of the effects caused by the electric field.

5. Further comments

This paper deals entirely with the effect of an electric field on the convection and heat transfer in a paraelectric gas. It is evident that substantial electric fields are required to produce observable changes in flow and heat transfer characteristics. However, the effect of an electric field on heat transfer may be even more apparent for fields of smaller intensity, in the case of a dielectric liquid. This would follow from the fact that for polar liquids the dielectric constant and its temperature dependence are large compared with those for a paraelectric gas. A possible approach for obtaining quantitative information on the effect of an electric field on heat transfer in a dielectric liquid would be to use empirical relationships for the variation of the density and dielectric susceptibility with temperature to replace expressions (2.7) and (2.12), respectively; also it might be necessary to revise the simplified equations (2.16) for the calculation of the electric field. This approach would describe body motions in the liquid as a direct consequence of the defects in the gravitational and electrical body forces produced by heating. It appears, however, that body motions can be produced in a dielectric liquid by an electric field even in the absence of heating. Thus Avsec & Luntz (1937), using electric field strengths of up to 10 kV/cm, have observed steady cellular patterns of motion in light oils. For example if the gap between two concentric cylinders is filled with a dielectric liquid and the outer cylinder is earthed whilst the electrical potential of the inner cylinder is raised then at some critical potential difference a two dimensional toroidal motion of the liquid occurs. In a second experiment Avsec & Luntz (1937) observed that an electric field can produce steady cellular motions in the interface regions between two immiscible dielectric fluids. A quantitive study of this latter type of instability in surface electro-convection has been made by Malkus & Veronis (1961). It is evident that an understanding of the former type of internal electroconvection is essential in the development of any theory on heat transfer in dielectric liquids.

One further comment may be of interest. Ivey & Lee (1956) attempted to repeat the experiments of Schwarz (1949) using moist air with a view to constructing a continuous-reading hygrometer. No appreciable change in heat transfer was produced by the electric field. The explanation is likely to be that H_2O molecules tend to associate in groups having an effective zero dipole moment (see Le Fèvre 1948).

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